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COMMENT

An alternative description of systems with a finite number of states

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Abstract. We will show how one can give a quantum description for systems having $(2n+1)$-dimensional anticommuting phase space. The Hilbert space related to these systems has 2^n dimensions. Among the applications of those systems are spin in three dimensions and the d-dimensional relativistic spinning particle.

1. Introduction

The standard way to describe the state of a spin is to use spinors and Pauli matrices [1, 2]. Here, we will show an alternative for this representation. In the usual description by means of a wavefunction, the arguments of this function are the polarised set of the phase-space variables, that is the maximal independent and commuting (with grading) set of variables. Usually, a bosonic phase space has an even dimension and every coordinate has a canonical conjugate momentum. However, in the fermionic case there could be a situation where a coordinate can be its own canonical conjugate, and the result is the possibility of an odd-dimensional phase space. We will show how one can properly polarise systems with an odd-dimensional fermionic phase space, and describe the state by a wavefunction. In these systems the states are eigenstates of an operator whose classical analogue is an even function of the phase-space variables.

2. Spin and fermionic coordinates

A simple example is spin in three dimensions, having the Lagrangian

$$L = \tfrac{1}{2}\mathrm{i}\xi\dot{\xi} - \tfrac{1}{2}\mathrm{i}B_k\varepsilon_{ijk}\xi_i\xi_j \qquad \{\xi_i, \xi_j\} = 0. \tag{1}$$

This Lagrangian describes a classical spin in a magnetic field $\boldsymbol{B}$. Poisson brackets will be defined in such a way that the graded commutator that resulted from them after quantisation will give the operator that corresponds to their result. Let the phase space be M with dimension D. The result of [1] and [2]†

$$\{f(\xi), g(\xi)\}_{\mathrm{PB}} = \mathrm{i}\sum_i (f(\xi)\overleftarrow{\partial}_{\xi_i})(\overrightarrow{\partial}_{\xi_i} g(\xi)) \tag{2}$$

† The result of [1] is obtained from the definition of [2] by using Dirac brackets that takes into account the second-class constraints of the first-order Lagrangian.

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is exact for polynomials of degree 2 at most in the ξs. This is in disagreement with the quantum commutators, for example:

$$\{\xi_1\xi_2\xi_3, \xi_1\xi_2\xi_3\}_{\mathrm{PB}} = 0 \neq \mathrm{i}\{\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3, \hat{\xi}_1\hat{\xi}_2\hat{\xi}_3\} = -\tfrac{1}{4}\mathrm{i}.$$

Let the graded commutator be

$$\{AB, CB]$$

where A, B, C are monomials of the $\hat{\xi}$s (A and C have no common ξs). One can prove that this graded commutator is non-zero only if B is an odd monomial. To identify this result with Poisson brackets one has to include all the derivatives of the odd orders, not only the first order (this may lead to a revised variational calculation, using variations of higher order).

One can redefine the Poisson brackets in such a way that they will give the same algebra as graded commutators also for polynomials of cubic or higher degree:

$$\{F, G\}^*_{\mathrm{PB}} = 2\mathrm{i} \sum_{\{n|2n-1\leq D\}} \sum_{i_1<i_2<\ldots<i_{2n-1}} (\tfrac{1}{2})^{2n-1} (F\overleftarrow{\partial_{i_1}\partial_{i_2}\ldots\partial_{i_{2n-1}}})(\overrightarrow{\partial_{i_{2n-1}}\ldots\partial_{i_2}\partial_{i_1}}G). \tag{3}$$

After quantisation of the system described by (1) we find the following algebra for the quantised variables $\hat{\xi}$:

$$\{\hat{\xi}_i, \hat{\xi}_j\} = \delta_{ij}.$$

Let us define

$$\hat{\xi}_1 = \mathrm{i}\hat{\xi}_3(\theta + \partial_\theta) \qquad \hat{\xi}_2 = \hat{\xi}_3(\theta - \partial_\theta) \qquad \hat{\xi}_3 = \hat{\xi}_3 \tag{4}$$

where θ and ∂_θ anticommute with $\hat{\xi}_3$.

A wavefunction is a function of θ, $\Psi = \Psi(\theta) = a + b\theta$. The difficulty here is that the action of $\hat{\xi}_i$ on the wavefunction is not defined in this representation. We have a well defined representation for bilinear functions, so this representation is restricted only to this kind of observable. The reason that for a three-dimensional phase space we can use only one variable for describing the state is that one can always rotate the system and make the interaction independent on one of ξ; so one variable can be decoupled.

The inner product is defined as

$$\langle\Psi|\Phi\rangle = \int \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, \bar{\Psi}(\bar{\theta})\Phi(\theta)\, \mathrm{e}^{\bar{\theta}\theta} \Rightarrow \langle\Psi|\Psi\rangle = |a|^2 + |b|^2.$$

The Hamiltonian is

$$H = \tfrac{1}{2}\mathrm{i}B_1(\theta - \partial) + \tfrac{1}{2}B_2(\theta + \partial) + B_3(\theta\partial - \tfrac{1}{2}). \tag{5}$$

After substitution and comparing powers of θ, one finds:

$$H\Psi = E\Psi \Rightarrow E^2 = \tfrac{1}{4}(B_1^2 + B_2^2 + B_3^2). \tag{6}$$

When $\boldsymbol{B} = B_3$,

$$\Psi_3^- = 1 \qquad E = -\frac{B_3}{2} \qquad \Psi_3^+ = \theta \qquad E = \frac{B_3}{2}.$$

When $\boldsymbol{B} = B_1$,

$$\Psi_1^\pm = \frac{1}{\sqrt{2}}(1 \pm \mathrm{i}\theta) \qquad E = \pm\frac{B_1}{2}.$$

One can check that the inner product between a eigenfunction of a Hamiltonian that is related to one direction of magnetic field with an eigenfunction that related to another, is the same as the usual treatment with Pauli matrices, with the equivalence: $\sigma_k \rightarrow \mathrm{i}\varepsilon_{ijk}\hat{\xi}_i\hat{\xi}_j$.

On the classical level Fermi parity is well defined, odd and even parities are related to odd and even powers of ξ. However, in this representation on the quantum level there is no meaning for this parity; generators of rotation, for example, can be even or odd powers of θ (classically they are even observables); $\frac{1}{2}(\theta+\partial_\theta) = \mathrm{i}\hat{\xi}_1\hat{\xi}_3$, $\frac{1}{2}\mathrm{i}(\theta-\partial_\theta) = \mathrm{i}\hat{\xi}_3\hat{\xi}_2$ and $\theta\partial_\theta - \frac{1}{2} = \mathrm{i}\hat{\xi}_2\hat{\xi}_1$ are all even functions on the classical level.

One observes here that this system has two states and one can use this result for describing classically a two-state system. Let the quantum states be: $|+\rangle$, $|-\rangle$, which means that every superposition that is transformed by the group SU(2) is also a state. There are two ways to describe the generators of SU(2).

(i) $G_1 = \hat{\psi}_1/\sqrt{2}$, $G_2 = \hat{\psi}_2/\sqrt{2}$, $G_3 = \mathrm{i}\hat{\psi}_1\hat{\psi}_2$ where: $\{\hat{\psi}_i, \hat{\psi}_j\} = \delta_{ij}$. This method describes the well known Fermi oscillator [3]. On the classical level, however, one will observe a difficulty in identifying these generators with classical observables. Transformations of the phase-space functions are generated by Poisson brackets (PB), and the commutator between $\hat{\psi}_1$ and $\hat{\psi}_2$ has no classical analogue because the PB between anticommuting objects becomes a $-\mathrm{i}\times$ *anticommutator* after quantisation; therefore, there is no PB classical analogue for the SU(2) algebra.

(ii) $G_1 = \mathrm{i}\hat{\xi}_2\hat{\xi}_3$, $G_2 = \mathrm{i}\hat{\xi}_3\hat{\xi}_1$, $\mathrm{i}\hat{\xi}_1\hat{\xi}_2$ where $\{\hat{\xi}_i, \hat{\xi}_j\} = \delta_{ij}$. Classically, the action of SU(2) is described as rotations on the ξ phase space.

One observes here that two classical systems describe the same quantum system, namely, the Fermi oscillator and the spin system, both having the same Hilbert space. It seems natural to describe a two-state quantum system by means of three anticommuting variables, as perhaps a better alternative for the standard Fermi oscillator description and the reason is a better understanding of the SU(2) action on the classical variables. The other description of a two-state system can be in parallel with the description of spin by means of bosonic compact phase space [4].

3. A relativistic, spin-$\frac{1}{2}$, charged, massive particle

In four dimensions, the states are superpositions of four states related to the spinning particle and antiparticle, in a given spatial momentum (namely:

$$\Psi = \alpha|\uparrow+\rangle + \beta|\downarrow+\rangle + \gamma|\uparrow-\rangle + \delta|\downarrow-\rangle$$

where $\uparrow$, $\downarrow$, $+$ and $-$ are spin up, spin down, particle and antiparticle respectively). The job of the anticommuting variables is to put the wavefunction on a four-dimensional column, and this can be done by using two variables, θ_1 and θ_2.

The phase-space contains five anticommuting variables, ξ_μ, ξ_5. On the quantum level, these variables obey the algebra: $\{\hat{\xi}_\mu, \hat{\xi}_\nu\} = \eta_{\mu\nu}$, $\{\hat{\xi}_\mu, \hat{\xi}_5\} = 0$, $\hat{\xi}_5^2 = \frac{1}{2}$, and one can realise them as the following:

$$\begin{aligned} &\hat{\xi}_1 = \mathrm{i}\hat{\xi}_5(\partial_1 + \theta_1) \qquad \hat{\xi}_2 = \hat{\xi}_5(\partial_1 - \theta_1) \\ &\hat{\xi}_3 = \mathrm{i}\hat{\xi}_5(\partial_2 + \theta_2) \qquad \hat{\xi}_0 = \mathrm{i}\hat{\xi}_5(\partial_2 - \theta_2) \qquad \hat{\xi}_5 = \hat{\xi}_5 \\ &\{\theta_i, \theta_j\} = \{\theta_i, \hat{\xi}_5\} = \{\partial_i, \partial_j\} = \{\partial_i, \hat{\xi}_5\} = 0. \end{aligned} \tag{7}$$

One can observe that the analogue of the γ matrices are the following: $\gamma_\mu = 2\mathrm{i}\hat{\xi}_\mu\hat{\xi}_5$. The Dirac equation can be expressed by those variables as follows:

$$
\begin{aligned}
&(P^\mu 2\mathrm{i}\hat{\xi}_\mu\hat{\xi}_5 + m)\Psi = 0 \\
&P^\mu\gamma_\mu = -\mathrm{i}\partial_{x^1}(\partial_{\theta_1} + \theta_1) + \partial_{x^2}(\partial_{\theta_1} - \theta_1) - \mathrm{i}\partial_{x^3}(\partial_{\theta_2} + \theta_2) - \mathrm{i}\partial_{x^0}(\partial_{\theta_2} - \theta_2) \\
&\Psi = \psi_0(x) + \psi_1(x)\theta_1 + \psi_2(x)\theta_2 + \psi_{12}(x)\theta_1\theta_2 .
\end{aligned}
\tag{8}
$$

Because of charge conservation†

$$|\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2 = \textit{conserved}$$

the most general transformations group is SU(2, 2). The group SU(2, 2) has fifteen generators that obey commutation relations (having even parity). The inner product that describes the group SU(2, 2) is

$$\langle\Psi|\Phi\rangle = \int \mathrm{d}\theta_1\,\mathrm{d}\bar{\theta}_1\,\mathrm{d}\theta_2\,\mathrm{d}\bar{\theta}_2\,\tilde{\Psi}(\bar{\theta}_1, \bar{\theta}_2)\Phi(\theta_1, \theta_2)\,\mathrm{e}^{\bar{\theta}_1\theta_1 - \bar{\theta}_2\theta_2}$$

where $\tilde{}$ reverses the order in every monomial and takes a complex conjugate of its coefficient.

The fifteen generators have the algebra of the conformal group (that is isomorphic locally to SU(2, 2)) by the following identifications‡:

$$
\begin{aligned}
&\Pi_i, (K_i) = \mathrm{i}\hat{\xi}_i\hat{\xi}_5 + (-)\varepsilon_{ijk}\hat{\xi}_j\hat{\xi}_k\hat{\xi}_0\hat{\xi}_5 \qquad \Pi_0, (K_0) = \mathrm{i}\hat{\xi}_0\hat{\xi}_5 + (-)2\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3\hat{\xi}_5 \\
&\Delta = 2\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3\hat{\xi}_0 \qquad M_{\mu\nu} = \mathrm{i}\hat{\xi}_\mu\hat{\xi}_\nu .
\end{aligned}
\tag{9}
$$

4. The massless fermion

This system is characterised by the constraint:

$$(\mathrm{i}\gamma_1\gamma_2\gamma_3\gamma_0 \pm 1)\Psi_\pm \equiv (4\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3\hat{\xi}_0 \pm 1)\Psi_\pm = 0. \tag{10}$$

Substituting the representation with the θ, one arrives at the following:

$$(4\theta_1\partial_1\theta_2\partial_2 - 2\theta_1\partial_1 - 2\theta_2\partial_2)\Psi_- = 0 \Rightarrow \Psi_- = \psi_0 + \psi_{12}\theta_1\theta_2 \tag{11a}$$

or

$$(2 + 4\theta_1\partial_1\theta_2\partial_2 - 2\theta_1\partial_1 - 2\theta_2\partial_2)\Psi_+ = 0 \Rightarrow \Psi_+ = \psi_1\theta_1 + \psi_2\theta_2 . \tag{11b}$$

Note that the constraint reduces the phase space by two variables, and as expected the reduced phase space is three dimensional as needed for describing the two-state system.

These are examples where the polarisation cuts the phase space M to be a space with dimension less than a half of $\dim(M)$. For a relativistic spinning particle in dimension d, one can use $[d/2] = n$ Grassmann variables for describing the state, and the Hilbert space is 2^n dimensional. That means that the phase space has to be $2n+1$ dimensional as will be explained below.

† We exemplify an unphysical situation where the particle is constrained to have one momentum. The right charge conservation is related to $\mathrm{SU}(\infty, \infty)$.

‡ The conformal algebra is valid on the classical level with the identification of PB by (3).

Expressions for the γ matrices are

$$d=2n \qquad \gamma_\mu = 2\mathrm{i}\hat{\xi}_{2n+1}\hat{\xi}_\mu \tag{12a}$$

$$d=2n+1 \qquad \gamma_\mu = \frac{(2\mathrm{i})^n}{(2n)!}\varepsilon_{i_1,i_2,\ldots\mu,\ldots i_{2n}}\hat{\xi}_{i_1}\hat{\xi}_{i_2}\ldots\hat{\xi}_{i_{2n}} \qquad i_j \neq \mu. \tag{12b}$$

These expressions for the γ matrices obey the Clifford algebra, as is needed for the relation between Dirac and Klein-Gordon equations. An interesting point is the possibility to have a one-particle description of systems with internal symmetry [5], by using anticommuting phase-space variables [6], such that after quantisation the state is a function of the polarised set of n variables.

5. Generalisation

For a general system that has 2^n states one observes that the wavefunction that depends on n anticommuting variables is transformed by the $\mathrm{SU}(2^n)$ group, generated by $2^{2n}-1$ generators. The classical analogue of these generators are observables that are polynomials on the ξ space. Here we will show two methods for describing these generators.

Method 1. Making use of $2n$ Grassmann variables (ξ_k on the classical level and $\hat{\xi}_k$ on the quantum level), for describing all the generators one will use all the powers of ξ, namely:

$$2^{2n}-1=\binom{1}{2n}+\binom{2}{2n}+\ldots+\binom{2n}{2n}.$$

One can easily see that the Poisson brackets between two odd monomials fails to describe $\mathrm{SU}(2^n)$ properly because it turns out to be an anticommutator on the quantum level.

Method 2. The phase-space M contains $2n+1$ Grassmann variables. The $\mathrm{SU}(2^n)$ generators are built by all the *even* powers of ξ. The number of these generators is

$$2^{2n}-1=\binom{2}{2n+1}+\binom{4}{2n+1}+\ldots+\binom{2n}{2n+1}.$$

One can argue that this method gives a correct relation between quantum operators and classical observables, by the representation

$$\hat{\xi}_{2i-1}=\mathrm{i}\hat{\xi}_{2n+1}(\theta_i+\partial_i) \qquad \hat{\xi}_{2i}=\hat{\xi}_{2n+1}(\theta_i-\partial_i)$$

and its inverse

$$\theta_i=-\mathrm{i}\hat{\xi}_{2n+1}\hat{\xi}_{2i-1}+\hat{\xi}_{2n+1}\hat{\xi}_{2i} \qquad \partial_i=-\mathrm{i}\hat{\xi}_{2n+1}\hat{\xi}_{2i-1}-\hat{\xi}_{2n+1}\hat{\xi}_{2i}.$$

This gives a one-to-one correspondence amongst operators of $\hat{\xi}$.

Related to this formalism, one can find a description of the $\mathrm{SU}(N)$ group when N is not restricted to be 2^n. One can describe the state as a function of $2(N-1)$ anticommuting variables $\theta_1, \bar{\theta}_1, \ldots \theta_{N-1}, \bar{\theta}_{n-1}$ with the restriction

$$f(\theta_1,\bar{\theta}_1,\ldots\theta_{N-1},\bar{\theta}_{N-1})=f(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2+\ldots+\bar{\theta}_{N-1}\theta_{N-1})\equiv f(\theta).$$

The $\mathrm{SU}(N)$ generators can be expressed by the basis $\theta^n\partial^m$, n, $m=0,1,2,\ldots N-1$ (excluding $\theta^0\partial^0$), where $\partial\equiv\Sigma_k\,\partial_k\bar{\partial}_k$. This restriction can result by imposing some constraints [6].

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